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A New Modification of the Rojo Method for Solving Symmetric Circulant Five-Diagonal Systems of Linear Equations

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Abstract—A new, effective, and stable modification of the Rojo method [1] for solving of real symmetric circulant five-diagonal systems of linear equations is proposed. This special kind of system appears in many applications: spline approximation, difference solution of partial differential equations, etc. The method presented in this paper a very efficient and stable method. © 1900 Elsevier Science Ltd. All rights reserved.

Keywords—Circulant linear system, Diagonal dominant matrix, Five-diagonal circulant matrix, Toeplitz matrix, Woodbury formula.

1. INTRODUCTION

Many problems in mathematics lead to the solution of sparse linear systems. Such problems are the solution of elliptic difference equations by finite difference or finite element methods [2]. When we consider the finite difference approximation of Poisson's equation on a rectangular region with Dirichlet boundary conditions we obtain the systems with a block tridiagonal matrix [2–4] or a band matrix [3,4]. In this problem, when we replace the Dirichlet boundary conditions by periodicity conditions the matrix of a resulting linear system is a block circulant matrix [3,4]. This system reduces the problem to the solution of n circulant linear systems which may be solved in parallel. In the solution of one-dimensional elliptic equations subject to periodic boundary conditions and the approximation of periodic functions using splines, banded circulant matrices are encountered [5]. These circulant matrices can be factored as a product of two simpler circulants [4] and the systems may then be solved by using the Woodbury formula [6].

The direct methods for solution of this special banded linear systems are very effective ones [2,4,7,8]. They are attractive since in theory they yield the exact solution. There are the direct methods which are based on a LU-decomposition [9] and direct methods which are based on the use of the fast Fourier transform [3,7,10].

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In [1] Rojo proposes a new stable method for solving of real symmetric tridiagonal circulant linear algebraic systems of equations. Here we extend this method for solving of real symmetric five-diagonal circulant $n \times n$ linear systems. Our method uses a LU-decomposition and requires $O(n)$ operations. This method is competitive with Gaussian elimination in terms of arithmetic operations and storage requirements. Such kind of systems have the form

$$Mx = f, \tag{1.1}$$

where

$$M = \begin{pmatrix} a & b & c & & & & & c & b \\ b & a & b & \cdot & & & & & c \\ c & b & a & \cdot & \cdot & & 0 & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & 0 & & & \cdot & \cdot & \cdot & \cdot & \cdot \\ c & & & & & c & b & a & b \\ b & c & & & & & c & b & a \end{pmatrix} = M(a,b,c,0,\dots,0,c,b) \tag{1.2}$$

is $n \times n$ matrix ($n \geq 5$). Further, we assume, that M is strongly diagonal dominant, i.e.,

$$|a| > 2|b| + 2|c|, \tag{1.3}$$

where $c \neq 0$. (The case $c = 0$ is considered by Rojo in [1].) Further, without loss of generality, we assume that $a > 0$, $c = -1$, or $a > 0$, $c = 1$. Then for these two cases (1.3) takes the form

$$a > 2m + 2,$$

where $m = |b|$.

2. SYMMETRIC THREE-PARAMETRIC FIVE-DIAGOANAL LINEAR SYSTEMS

In this section, we shall study the three-parametric linear systems

$$Ny = f, \tag{2.1}$$

where $n \times n$ matrix $N = (n_{ij})$ has the form

$$N = N(a,b,c;\alpha,\beta,\gamma) = \begin{pmatrix} \alpha & \beta & c & & & & & \\ \beta & \gamma & b & \cdot & & & & \\ c & b & a & \cdot & \cdot & & 0 & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot & \cdot & c \\ & 0 & & & \cdot & \cdot & \cdot & b \\ & & & & & c & b & a \end{pmatrix},$$

i.e.,

$$n_{ij} = n_{ji} = \begin{cases} \alpha, & i = j = 1, \\ \beta, & i = 1, \quad j = 2, \\ \gamma, & i = j = 2, \\ a, & i = j > 2, \\ b, & i - j = 1, \quad j > 1, \\ c, & i - j = 2, \quad j > 2, \\ 0, & \text{in the other cases.} \end{cases}.$$

The problem is to find the parameters α, β, γ, in such a way that N to have a real LU-factorization

$$N = LU. \tag{2.2}$$

CASE 1. $c = -1$ ($\delta^2 = a - 2m - 2 > 0, m > 0$).

In this case, we find L and U in LU-factorization (2.2) of the form

$$L = \begin{pmatrix} \alpha & & & & & & \\ \beta & \alpha & & & & 0 & \\ -1 & \cdot & \cdot & & & & \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & 0 & & & \cdot & \cdot & \cdot \\ & & & & -1 & \beta & \alpha \end{pmatrix},$$

$$U = \begin{pmatrix} 1 & \frac{\beta}{\alpha} & -\frac{1}{\alpha} & & & & \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & 0 & \\ & & & \cdot & \cdot & \cdot & \\ & & & & \cdot & \cdot & \cdot \\ & & & & & \cdot & -\frac{1}{\alpha} \\ & 0 & & & & \cdot & \frac{\beta}{\alpha} \\ & & & & & & 1 \end{pmatrix},$$

i.e.,

$$N = \frac{1}{\alpha} L L^{\top} = \alpha U^{\top} U.$$

The last equations are equivalent the nonlinear system

$$\begin{aligned} \gamma + \frac{1}{\alpha} &= a, \\ \beta - \frac{\beta}{\alpha} &= b, \\ \alpha + \frac{\beta^2}{\alpha} &= \gamma. \end{aligned} \tag{2.3}$$

We are interested in a solution in real numbers. From (2.3) by eliminating of β and γ, we receive

$$\left[\alpha m - (1-\alpha)^2\right]^2 = \alpha(1-\alpha)^2\delta^2 > 0. \tag{2.4}$$

It is easy to see, that if (α, β, γ) is a real solution of (2.3), then $\alpha > 0$ and $\alpha \neq 1$. Transforming (2.4) into

$$\frac{m\sqrt{\alpha}}{1-\alpha} - \frac{1-\alpha}{\sqrt{\alpha}} = \pm\delta \tag{2.5}$$

and putting

$$\frac{1-\alpha}{\sqrt{\alpha}} = x.$$

From (2.5), we get four values for x:

$$x_i = \frac{\pm\delta \pm \sqrt{\delta^2 + 4m}}{2}.$$

We arrange them as follows:

$$x_1 < x_2 < 0 < x_3 < x_4.$$

The corresponding $\alpha_i, \beta_i, \gamma_i$ are

$$\begin{aligned} \alpha_i &= \frac{-x_i \pm \sqrt{x_i^2+4}}{2}, \\ \beta_i &= \frac{b\alpha_i}{\alpha_i - 1}, \\ \gamma_i &= \alpha_i + \frac{\beta_i^2}{\alpha_i}, \end{aligned} \tag{2.6}$$

where

$$\alpha_1 > \alpha_2 > 1 > \alpha_3 > \alpha_4 > 0.$$

Hence, there are four possible factorizations of N

$$N = \frac{1}{\alpha_i} L_i L_i^\top, \qquad (i = 1, 2, 3, 4),$$

corresponding to the four solutions (2.6). Let us note the three extreme cases.

Case 1.1. $\delta > 0,\ m = 0\ (a > 2)$

$$\begin{array}{lcccc} \alpha_i: & \dfrac{a+\sqrt{a^2-4}}{2} & 1 & 1 & \dfrac{a+\sqrt{a^2-4}}{2}, \\ \beta_i: & 0 & -\sqrt{a-2} & \sqrt{a-2} & 0, \\ \gamma_i: & \dfrac{a+\sqrt{a^2-4}}{2} & a-1 & a-1 & \dfrac{a-\sqrt{a^2-4}}{2}. \end{array}$$

There are four different factorizations.

Case 1.2. $\delta = 0,\ m > 0\ (a = 2m + 1)$.

$$\begin{aligned} \alpha_1 = \alpha_2 &= \left(\frac{\sqrt{m+4}+\sqrt{m}}{2}\right)^2, \\ \alpha_3 = \alpha_4 &= \left(\frac{\sqrt{m+4}-\sqrt{m}}{2}\right)^2, \\ \beta_i &= \frac{\alpha_i\, b}{\alpha_i - 1}, \qquad i = 1, 2, 3, 4. \end{aligned}$$

There are two different factorizations.

Case 1.3. $\delta = m = 0,\ (a = 2):$

$$(\alpha, \beta, \gamma) = (1, 0, 1).$$

There is one factorization.

Case 2. $c = 1\ (a > 2m + 2,\ m > 0)$.

In this second case, the nonlinear system for α, β, γ has the form

$$\begin{aligned} \gamma + \frac{1}{\alpha} &= a, \\ \beta + \frac{\beta}{\alpha} &= b, \\ \alpha + \frac{\beta^2}{\alpha} &= \gamma. \end{aligned} \tag{2.7}$$

Again it is easy to see that if (α, β, γ) is a real solution of (2.7), then $\alpha > 0$. Now the elimination β and γ gives

$$\alpha + \frac{1}{\alpha} + \frac{\alpha m^2}{(\alpha+1)^2} = a, \tag{2.8}$$

after the substitution

$$\alpha + \frac{1}{\alpha} = x,$$

we transform (2.8) into quadratic equation for x with real and different solutions x_1, x_2 $(x_1 > x_2)$

$$x_{1,2} = \frac{a - 2 \pm \sqrt{(a+2)^2 - 4m^2}}{2}. \tag{2.9}$$

For every one solution x_i, we get two values α_{ij}:

$$\alpha_{i,j} = \frac{x_i \pm \sqrt{x_i^2 - 4}}{2}.$$

When $\alpha_{i,1}$ and $\alpha_{i,2}$ are real, we suppose that $\alpha_{i,1} \geq \alpha_{i,2}$. It is clear that for i fixed, $\alpha_{i,j}$ will be a component of the solution of our problem, if $x_i \geq 2$. We shall show that the last condition is always satisfied for x_1 from (2.9). More exactly we shall show, that $x_1 \geq 4$. Indeed

$$\begin{aligned} x_1 &= \frac{a - 2 + \sqrt{(a+2)^2 - 4m^2}}{2} \\ &> \frac{2m + \sqrt{(2m+4)^2 - 4m^2}}{2} \\ &= m + 4\sqrt{m+1} > m + 4 > 4. \end{aligned}$$

Then

$$\alpha_{1,1} > \frac{2m+7}{2} > 1, \qquad \alpha_{1,2} < \frac{2}{2m+7} < 1.$$

For x_2 from (2.9) one can prove that $x_2 \geq 2$, if $a > 18$ and $m \in [2\sqrt{a-2}, (a-2)/2]$.

So in this case N can have $2, 3$, or 4 LU factorizations.

REMARK 1. For the case $c = 1$, we can mark three extreme possibilities, too.

REMARK 2. If for $(\alpha > 0, \beta, \gamma)$ there exists LU factorization of N, the matrix N is positive definite.

DEFINITION 1. *A solution (α, β, γ) and corresponding LU factorization of N are optimal when they correspond to $\alpha = \alpha_{\max}$.*

THEOREM 1. *If (α, β, γ) optimal, then*

$$\begin{array}{ll} \text{(a)} & a - 2m - 2 < \gamma - |\beta| - m - 1 < \alpha - |\beta| - 1, \\ \text{(b)} & a > \gamma > \alpha, \\ \text{(c)} & a + 2m + 2 > \gamma + |\beta| + m + 1 > \alpha + |\beta| + 1. \end{array} \tag{2.10}$$

PROOF. Case $c = -1$. As we saw, in this case $\alpha_1 = \alpha_{\max}$, correspond to $x_1 = x_{\min}$

$$x_1 = \frac{-\delta - \sqrt{\delta^2 + 4m}}{2}.$$

From the last expression we get $m = x^2 + \delta x$, i.e.,

$$m < x^2. \tag{2.11}$$

The inequality (2.11) is equivalent to

$$\frac{1}{\alpha} + \frac{m}{\alpha - 1} < 1,$$

from which we find

$$\begin{aligned} \gamma - m - |\beta| - 1 &= a - \frac{1}{\alpha} - m - \frac{m\alpha}{\alpha - 1} - 1 \\ &= a - 2m - 1 - \left(\frac{1}{\alpha} + \frac{m}{\alpha - 1}\right) \\ &> a - 2m - 1 - 1 = a - 2m - 2. \end{aligned}$$

The second inequality of (2.10a) is true too, since it is equivalent of (2.11).

The inequalities (2.10b,c) follow immediately. Namely

$$\begin{aligned} a = \gamma + \frac{1}{\alpha} > \gamma &= \frac{\beta^2}{\alpha} + \alpha, \\ (a + 2m + 2) - (\gamma + |\beta| + m + 1) &= \frac{1}{\alpha} + m + 1 - |\beta|, \\ \frac{1}{\alpha} + m + 1 - \frac{m\alpha}{\alpha - 1} &= \frac{1}{\alpha} + 1 - \frac{m\alpha}{\alpha - 1} \\ &> \frac{1}{\alpha} + 1 + \frac{1}{\alpha} - 1 = \frac{2}{\alpha} > 0, \\ (\gamma + |\beta| + m + 1) - (\alpha + |\beta| + 1) &= \frac{\beta^2}{\alpha} + m > 0. \end{aligned}$$

Case $c = 1$.

$$\begin{aligned} \gamma - |\beta| - m - 1 &= a - \frac{1}{\alpha} - |\beta| - m - 1 \\ &= a - \frac{1}{\alpha} - \frac{m\alpha}{\alpha + 1} - m - 1 > a - 2m - 2. \end{aligned}$$

So we proved (2.10a).

As for the second inequality of (2.10a), it is true since it is easy to see that it is equivalent to

$$\frac{\alpha m^2}{(\alpha + 1)^2} + \alpha > m.$$

But the last inequality is equivalent to $a > m + 1/\alpha$, which is true. So the second inequality of (2.10a) is true too.

The inequalities (2.10b,c) one can proved, as in the case $c = -1$.

COROLLARY 1. *If $N_{\max} = N(a, b, -1; \alpha_{\max}, \beta, \gamma) = (1/\alpha_{\max}) L_{\max} L_{\max}^{\top}$, then $N_{\max}$ and $L_{\max}$ are strongly diagonal matrices.*

REMARK 3. In the above mentioned three extreme cases, some of the inequalities (2.10) become equalities.

Let $A = (a_{ij})$ be a $n \times n$ strongly dominant diagonal matrix, i.e.,

$$r_i = |a_{ii}| - \sum_{j \neq i} |a_{ij}| > 0, \qquad i = 1, \dots, n.$$

For such kind of matrices there are known the following results for A^{-1} and its elements a'_{ij} [11]:

$$\|A^{-1}\|_{\infty} \leq \max_k \frac{1}{r_k}, \tag{2.12}$$

$$|a'_{ij} a_{jj}| \leq \left(\min_k \frac{r_k}{|a_{kk}|}\right)^{-1} \left(1 - \min_k \frac{r_k}{|a_{kk}|}\right)^{|i-j|/s}. \tag{2.13}$$

In (2.13), we suppose that A is $(2s+1)$-band matrix, i.e., $a_{ij} = 0$ for $|i-j| > s > 0$. Let

$$N^{-1} = N_{\max}^{-1} = \left(n'_{ij}\right), \quad L^{-1} = L_{\max}^{-1} = \left(\alpha'_{ij}\right) = \left(\alpha'_{i-j}\right),$$

where $\alpha_{ij} = 0$ for $i < j$. Then from (2.10), (2.12), and (2.13) we obtain

$$\begin{aligned}
\|N^{-1}\|_\infty &\le \frac{1}{a-2m-2} = \frac{1}{\delta}, \\
\|L^{-1}\|_\infty &\le \frac{1}{\alpha - |\beta| - 1} < \frac{1}{\delta}, \\
\operatorname{cond}_\infty(N) = \|N\|_\infty \|N^{-1}\|_\infty &< \frac{a+2m+2}{a-2m-2}, \\
\operatorname{cond}_\infty(L) = \|L\|_\infty \|L^{-1}\|_\infty &< \frac{\alpha+|\beta|+1}{\alpha-|\beta|-1} < \operatorname{cond}_\infty(A), \\
\left|n'_{ij}\right| &< \frac{a}{2\delta}\left(\frac{2m+2}{a}\right)^{|i-j|/2}, \\
|\alpha'_k| &< \frac{\alpha}{\alpha-|\beta|-1}\left(\frac{|\beta|+1}{2}\right)^{k/2}.
\end{aligned}$$

From these two inequalities it follow that $n'_{ij} \to 0$ and $\alpha'_k \to 0$ when $|i-j|$ and k increase, respectively. The relation $\alpha'_k \to 0$ $(k \to \infty)$ can be obtained directly from the formula

$$\alpha'_k = \frac{1}{\Delta}\left[\left(\frac{\Delta-\beta}{2\alpha}\right)^{k+1} + (-1)^k\left(\frac{\Delta+\beta}{2\alpha}\right)^{k+1}\right],$$

for the elements of L^{-1}, where $\Delta = \sqrt{\beta^2 + 2\alpha}$.

Having the LU factorization of N, the solution of the system $Ny = f$, i.e., $LUy = f$ we can be reduced to the solution of linear systems: $Lz = f$; $Uy = z$.

3. SOLUTION OF FIVE-DIAGONAL SYMMETRIC TOEPLITZ LINEAR SYSTEM

Here, we shall apply the results of Section 2 for solving symmetric Toeplitz linear system of the form

$$Pu = f, \tag{3.1}$$

where the symmetric Toeplitz matrix P has the form

$$P = \begin{pmatrix}
a & b & c & & & & \\
b & a & b & \cdot & & & \\
c & b & a & \cdot & \cdot & 0 & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & \cdot & \cdot & \cdot & c \\
 & 0 & & \cdot & \cdot & \cdot & b \\
 & & & & c & b & a
\end{pmatrix} = P(a,b,c;a,b,a),$$

and $c = -1$ or $c = 1$, $a > 2m+2$. The solution of the system (3.1) begin with with solution the corresponding parametric linear system

$$Ny = f,$$

where $N = N(a,b,c;\, \alpha,\beta,\gamma)$. The parameters (α,β,γ) can be found as in Section 2. Then for the matrix

$$R = \begin{pmatrix} a-\alpha & b-\beta \\ b-\beta & a-\gamma \end{pmatrix} = \frac{1}{\alpha}\begin{pmatrix} \beta^2+1 & \pm\beta \\ \pm\beta & 1 \end{pmatrix},$$

we use the factorization

$$R = SS^\top,$$

where

$$S = \frac{1}{\sqrt{\alpha}}\begin{pmatrix} \sqrt{\beta^2+1} & 0 \\ \pm\dfrac{\beta}{\sqrt{\beta^2+1}} & \dfrac{1}{\sqrt{\beta^2+1}} \end{pmatrix}. \tag{3.2}$$

In this case, the matrices P and N are connected by the relation

$$P = N + UU^\top, \tag{3.3}$$

where

$$U = \begin{pmatrix} S \\ 0 \end{pmatrix},$$

and 0 is the $(n-2)\times 2$ zero matrix. Using (3.3) and the Woodbury's formula [6], we receive

$$\begin{aligned} P^{-1} &= N^{-1} - N^{-1}U\left(I + U^\top N^{-1}U\right)^{-1} U^\top N^{-1}, \\ u = P^{-1}f &= y - N^{-1}U\left(I + U^\top N^{-1}U\right)^{-1} U^\top y. \end{aligned} \tag{3.4}$$

It is clear that the solution u can be found from the last formula (3.4) by successive calculation of the expressions:

$$\begin{gathered} y = N^{-1}f,\ N^{-1}U,\ U^\top N^{-1}U,\ \left(I + U^\top\left(N^{-1}U\right)\right)^{-1}, \\ U^\top y, \quad z = \left(I + U^\top\left(N^{-1}U\right)\right)^{-1}\left(U^\top y\right),\ \left(N^{-1}U\right)z,\ \ u = P^{-1}f. \end{gathered}$$

The main part of above calculations is in finding the first two expressions, which is equivalent to solving of three linear systems of kind (2.1) with the same matrix N and different right-hand sides.

4. SOLVING OF FIVE-DIAGONAL SYMMETRIC CIRCULANT LINEAR SYSTEM

Now we can start with consideration of our new modification of the Rojo's method for solving $n\times n$ linear system of the kind (1.1),(1.2). This modification we shall consider only in the case $c=-1$, i.e., $a > 2m+2$. (The consideration of case $c=1$ is similar.) For this aim, let us introduce the notations:

$$\begin{aligned} &\hat f = (f_1, f_2, \dots, f_{n-2})^\top, && \tilde f = (f_{n-1}, f_n)^\top, && f = \begin{pmatrix} \hat f \\ \tilde f \end{pmatrix}, \\ &\hat x = (x_1, x_2, \dots, x_{n-2})^\top, && \tilde x = (x_{n-1}, x_n)^\top, && x = \begin{pmatrix} \hat x \\ \tilde x \end{pmatrix}, \\ &Q = \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}, && V^\top = \left(Q^\top, 0^\top, Q\right), && A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \end{aligned}$$

where 0 is $(n-6)\times 2$ zero matrix. According to above notations the system (1.1),(1.2) can be written in the form

$$\begin{pmatrix} P & V \\ V^\top & A \end{pmatrix}\begin{pmatrix} \hat x \\ \tilde x \end{pmatrix} = \begin{pmatrix} \hat f \\ \tilde f \end{pmatrix}, \tag{4.1}$$

where $(n-2)\times(n-2)$ matrix P has the form $P = P(a,b,-1;\ a,b,a)$. But (4.1) is equivalent to

$$\begin{aligned} P\hat{x} + V\tilde{x} &= \hat{f}, \\ V^{\mathsf{T}}\hat{x} + A\tilde{x} &= \tilde{f}. \end{aligned} \tag{4.2}$$

After elimination of $\tilde{x}$ from (4.2), we get the linear system

$$G\hat{x} = r,$$

where

$$G = P - VA^{-1}V^{\mathsf{T}}, \qquad r = \hat{f} - VA^{-1}\tilde{f}. \tag{4.3}$$

If we apply again the Woodbury's formula from the first relation (4.3), we obtain

$$G^{-1} = P^{-1} + P^{-1}V\left(A + V^{\mathsf{T}}P^{-1}V\right)^{-1}V^{\mathsf{T}}P^{-1}$$

and

$$\hat{x} = G^{-1}r = u + P^{-1}V\left(A + V^{\mathsf{T}}P^{-1}V\right)^{-1}V^{\mathsf{T}}u, \tag{4.4}$$

where $u = P^{-1}r$.

To compute $\hat{x}$ we use (4.4) in a similar way as (3.4). The difference here is that the main calculation is for solving of three $(n-2)\times(n-2)$ linear systems of kind (3.1) with the same matrix P and different right-hand sides.

Finding of $\hat{x}$ from (4.4), we get $\tilde{x}$ from the second equation (4.2) by formula

$$\tilde{x} = A^{-1}\left(\tilde{f} - V^{\mathsf{T}}\hat{x}\right). \tag{4.5}$$

5. NUMERICAL EXPERIMENTS

The method described here were tried for $n\times n$ symmetric circulant linear systems $Mx = f$ with exact solution $x = (1,1,\dots,1)^{\mathsf{T}}$ and

I. $M = M(n+1/2, n/2-1, -1, 0, \dots, -1, n/2-1)$,
II. $M = M(n+1, n/2-1, -1, 0, \dots, -1, n/2-1)$,
III. $M = M(n+2, n/2-1, -1, 0, \dots, -1, n/2-1)$.

The results $\tilde{x}$ are given in the next table in which $\varepsilon = \|x - \tilde{x}\|_\infty$.

Table 1.

n	$\varepsilon = \|x - \tilde{x}\|_\infty$		
	I	II	III
20	$1.0911E-11$	$1.2732E-11$	$9.0951E-12$
50	$9.0949E-11$	$4.1836E-11$	$1.2733E-11$
100	$1.4006E-10$	$1.9281E-10$	$7.8216E-11$
200	$2.5829E-9$	$3.1468E-10$	$2.8012E-10$
400	$2.3756E-9$	$2.2973E-9$	$3.2559E-10$
800	$1.7455E-8$	$2.8031E-9$	$2.3501E-9$
1000	$2.0372E-9$	$1.5643E-9$	$4.1836E-10$

6. GENERAL CONCLUSIONS AND REMARKS

The method described here is a very effective and stable one, provided optimal LU factorization is used.

Our method is competitive the other methods [4] for solving circulant linear systems which appear in many appliications.

The realization of the method need $O(n)$ operations.

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